Intersection and Disjointness Graphs of Convex Sets

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Abstract

We give a survey of results on disjointness and intersection graphs of geometric objects. We start by proving Ramsey-type results for the intersection graphs of convex sets in the plane. We first show that every intersection graph of n convex sets in the plane has a clique or independent set of size $n^{1/5}$. After that we show that if G is the intersection graph of n convex sets in the plane, then G or $\bar{G}$ has a bi-clique of size cn for a universal constant c. We then show that a geometric graph on n vertices with no $k+1$ disjoint edges has at most 2^9k^2n edges. Finally, we look at the colouring properties of disjointness graphs and prove that the family of disjointness graphs of segments in the plane is χ-bounded. We also show a generalization of this result to higher dimensions and discuss the χ-boundedness of related families of disjointness graphs.

Contents

1 Introduction

A classic result in Ramsey theory by Erdős and Szekeres [9] states that every graph on n vertices has a clique or an independent set of size at least $\frac{1}{2}\log n$. It is believed that graphs that achieve this bound asymptotically are 'random looking' and we can get a much better bound for structured families of graphs. For a graph G, let $\mathcal{F}(G)$ be the family of all graphs that do not contain G as an induced subgraph. Erdős and Hajnal [8] showed that for any graph G, there exists a constant depending only on G, say $c(G)$, such that every graph in $\mathcal{F}(G)$ has a clique or an independent set of size at least $e^{c\sqrt{\log n}}$. They raise the question of whether we can improve this lower bound to n^c, which remains an important open problem and is called the Erdős-Hajnal conjecture.

If the two classes of vertices in a bipartite graph are of the same size or their sizes differ by at most one then the graph is called balanced. A balanced complete bipartite graph is called a bi-clique. The conjecture of Erdős and Hajnal [8] mentioned above motivates the definition of the following Ramsey-type properties of a family of graphs $\mathcal{F}$.

1. $\mathcal{F}$ has the *weak Erdős-Hajnal property* if there exists a constant a depending only on $\mathcal{F}$, say $a(\mathcal{F})$, such that every graph on n vertices in $\mathcal{F}$ contains a clique or an independent set of size at least $n^{a(\mathcal{F})}$.

2. $\mathcal{F}$ has the *strong Erdős-Hajnal property* if there exists a constant b depending only on $\mathcal{F}$, say $b(\mathcal{F})$, such that for every graph G on n vertices in $\mathcal{F}$, either G has a bi-clique of size $b(\mathcal{F})n$ or the complement of G has a bi-clique of size $b(\mathcal{F})n$.

While it is not immediately clear from the definition why one of the properties is called 'weak' and the other 'strong', it is not difficult to show that if a hereditary family of graphs has the strong Erdős-Hajnal property then it also has the weak Erdős-Hajnal property. The interested reader can refer to Alon *et al.* [1] for a proof. We look at a particular hereditary family of graphs, which is the family of intersection graphs of convex sets in the plane.

1.1 Intersection and disjointness graphs

An *intersection graph* of a collection of sets is a graph which has a vertex corresponding to each set in the collection and two vertices are connected by an edge if the corresponding sets have a non-empty intersection. A *disjointness graph* is similarly defined, but with an edge between two vertices if the corresponding sets are disjoint.

We start by proving a Ramsey-type result for families of convex sets in the plane, where we show that given a family of n convex sets in the plane, we can always find $n^{1/5}$ of them which are pairwise intersecting or pairwise disjoint. We then show a Ramsey-type result for families of axis-parallel rectangles in the plane proved by Larman *et al.* [15], where we show that any family of n axis-parallel rectangles in the plane must have $\sqrt{\frac{n}{2\log n}}$ rectangles that are mutually intersecting or mutually disjoint. We then show that the family of convex sets in the plane has the strong Erdős-Hajnal property as shown by Fox e*et al.* [10].

After that we move on to results on disjointness of edges in geometric graphs. We describe a result proved by Pach and Tőröcsik [19] which shows that any geometric graph with n vertices and more than nk^4 edges must have $k+1$ pairwise disjoint edges. Tóth [23], also using partial orders, improved upon this result by showing that any geometric graph with n vertices and more than $n2^9k^2$ edges must have $k+1$ pairwise disjoint edges.

Finally we see some colouring properties of disjointness graphs in the form of some recent results by Pach *et al.* [18]. In particular we show that the disjointness graph of a family of lines in the d-dimensional projective space is χ-bounded, as is the family of segments in $\mathbb{R}^d$.

We give some preliminary results and definitions which will be used later.

1.2 Preliminaries and definitions

Given a graph G we denote its vertex set by $V(G)$ and given $W \subseteq V(G)$ we denote by $G[W]$ the subgraph induced by W. We denote the clique number of G by $\omega(G)$ and the independence number by $\alpha(G)$. $\chi(G)$ is the chromatic number of G. We say that a graph G is perfect if for every induced subgraph G' of G we have $\chi(G') = \omega(G')$. We denote by K_t the complete graph on t vertices, and by $K_{t,t}$ the complete bipartite graph with t vertices in both classes, in other words a bi-clique of size $2t$.

We describe some basics about partial orders and state Dilworth's theorem [6], which is one of the main tools used.

1.2.1 Partial orders

Definition 1.1 (Partial order)**.** A partial order is a set P with a binary relation $\preceq$, denoted by $(P, \preceq)$ such that for all a, b, c in P

1. $a \preceq a$ (Reflexive)
2. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (Transitive)
3. $a \preceq b$ and $b \preceq a$ implies $a = b$ (Anti-symmetric)

Let P be a partial order with the relation $\preceq$. We say that two elements a and b of P are comparable by $\preceq$ if either $a \preceq b$ or $b \preceq a$, otherwise we say that they are incomparable. Let $\{p_1, p_2, \dots, p_k\} = C \subseteq P$. We say that C is a chain in P if every pair of elements in C is comparable, and we say that C is an antichain in P if every pair of elements in C is incomparable.

We will make frequent use of the following theorem, due to Dilworth [6].

Theorem 1.2 (Dilworth's Theorem)**.** *If P is a finite partial order, the cardinality of the largest antichain in P is equal to the smallest number of chains needed to cover P.*

What we will use are the following simple corollaries of the above theorem

Corollary 1.3. *Let P be a finite partial order of size n and p be a positive integer less than n. Then P contains either a chain of size p or an antichain of size $\lceil \frac{n}{p} \rceil$.*

We can extend this to more than one partial order.

Corollary 1.4. *Let $P_1, P_2, \ldots P_r$ be partial orders defined on the same set S of size n. Then S has a subset of size at least $n^{\frac{1}{r+1}}$ which is either a chain in one of the partial orders or an anti-chain in all of the partial orders.*

The *comparability graph* of a partial order P is a graph G in which we have a vertex corresponding each element in P, and there is an edge between two vertices if they are comparable by the partial order. The *incomparability graph* of a partial order is the complement of the comparability graph, so that we have an edge between two vertices if they are not comparable by a partial order. We state another corollary of Dilworth's theorem which we will be using.

Corollary 1.5. *The comparability graph of a partial order is perfect.*

The *linear extension* of a partial order $(P, \preceq)$ is a total order $\leq$ on P such that if $a \preceq b$ then $a \leq b$. Every partial order is the intersection of all its linear extensions. The *dimension* of a partial order is the minimum number of linear extensions which it is an intersection of. We denote by $\mathcal{C}_2$ and $\mathcal{I}_2$ the family of comparability and incomparability graphs of 2-dimensional partial orders respectively. It is not difficult to show that $\mathcal{C}_2 = \mathcal{I}_2$ [4].

We describe some Turán-type results which will be used in the proof of Theorem 2.4. Turán-type problems are problems of the kind where we wish to determine $ex(n, H)$, the maximum number of edges a graph on n vertices can have without having a subgraph isomorphic to H. We extend this definition naturally to $ex_{\mathcal{I}_2}(n, H)$ as the maximum number of edges that the incomparability graph of a 2-dimensional partial order on n elements can have without containing a subgraph isomorphic to H. $ex_{\mathcal{C}_2}(n, H)$ is analogously defined. We use the following result due to Fox *et al.* [10] for 2-dimensional partial orders, in the proof of Theorem 2.4.

Theorem 1.6 (Fox *et al.* [10])**.** *For the comparability and incomparability graphs of 2-dimensional partial orders with no bi-cliques of size $2t$ we have*

$$ex_{\mathcal{I}_2}(n, K_{t,t}) = ex_{\mathcal{C}_2}(n, K_{t,t}) \leq 2(t-1)n - \binom{2t-1}{2}$$

for every $t \geq 2$ and $n \geq 2t - 1$.

We also use the following result by Tomon [22] which says that the graph formed by taking a union of comparability graphs of a finite number of partial orders has a bi-clique of size a constant fraction of n as long as the graph has sufficient number of edges. More precisely

Theorem 1.7 (Tomon [22])**.** *Suppose G is the union of comparability graphs of r partial orders. Then for any $0 < \epsilon < (1/4)^r$ if $|E(G)| \geq (\frac{1}{2} - \frac{1}{2^{r+1}} + \epsilon)n^2$ then G has a bi-clique of size $\frac{\epsilon n}{r 2^r}$.*

We can finally move on to giving the promised proofs for the Ramsey-type results mentioned previously.

2 Ramsey-type results for intersection graphs

In this chapter we prove some Ramsey-type results on intersection graphs of convex sets in the plane. One of the simplest such results is as follows. Given a family $\mathcal{I} = \{I_1, I_2, \ldots I_n\}$ of intervals on a line such that no point on the line is contained in more than p intervals in $\mathcal{I}$, we can decompose $\mathcal{I}$ into p disjoint sets such that each set has only pairwise disjoint intervals. This implies that the intersection graph of n intervals on a line has an independent set or clique of size at least $\sqrt{n}$. Pach [17] showed that this result can be extended to a family of convex bodies in $\mathbb{R}^d$ for which the ratio of the circumradius to the inradius of the convex bodies is bounded by a constant.

2.1 Erdős-Hajnal properties

The natural question which arises then is to find the largest number $f(n)$ such that the intersection graph of any family of n convex sets in the plane has an independent set or clique of size $f(n)$. We show that $f(n) \geq n^{1/5}$, or in other words, the weak Erdős-Hajnal property holds for the family of intersection graphs of convex sets in the plane.

2.1.1 Weak Erdős-Hajnal property

We give a simple proof using partial orders to show that $f(n)$ has the weak Erdős-Hajnal property.

Theorem 2.1 (Larman *et al.* [15])**.** *Let $f(n)$ be the largest number with the property that for any family of n compact convex sets in the plane, we have at least $f(n)$ of them to be pairwise disjoint or pairwise intersecting. Then $f(n) \geq n^{1/5}$.*

We note that we can assume without loss of generality that the sets are compact. Since we are only looking at intersection graphs of a finite number of convex sets, we can only have a finite number of intersecting pairs. We choose one point within the intersection of every intersecting pair and call this set of points special points. Therefore, we can find a bounding box which contains each of the special points. Clipping each of the convex sets to be inside this bounding box gives us a family of compact convex sets which has the same intersection graph.

We proceed to prove the theorem by defining four partial orders on compact convex sets in the plane.

Given a convex set C in the plane, we define by $x(C)$ the projection of C onto the x-axis. Given two convex sets C_1 and C_2 in the plane, we say that C_1 above C_2 if for every vertical line l which intersects both C_1 and C_2, every point in $l \cap C_1$ is above every point in $l \cap C_2$.

Definition 2.2 (Partial orders on convex sets in the plane)**.** For two convex sets C_1 and C_2 in the plane (Figure 2.1)

- $C_1 \preceq_1 C_2$ if $x(C_1) \subseteq x(C_2)$ and C_1 lies below C_2
- $C_1 \preceq_2 C_2$ if $x(C_1) \subseteq x(C_2)$ and C_1 lies above C_2

- $C_1 \preceq_3 C_2$ if C_1 is below C_2 and the right end point of $x(C_1)$ is the right of the right end point of $x(C_2)$ and the left end point of $x(C_1)$ is to the right of the left end point of $x(C_2)$
- $C_1 \preceq_4 C_2$ if C_1 is above C_2 and the right end point of $x(C_1)$ is the right of the right end point of $x(C_2)$ and the left end point of $x(C_1)$ is to the right of the left end point of $x(C_2)$

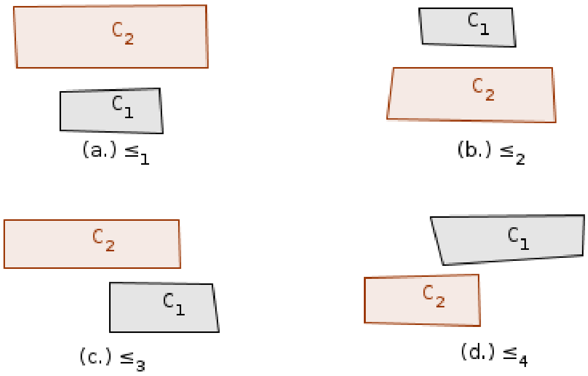

FIGURE 2.1: The partial orders defined on the convex sets in the plane

Proof of Theorem 2.1: It is not difficult to see that if two convex sets in the plane are disjoint, they must be comparable by at least one of these partial orders. So a chain in any of the partial orders corresponds to a pairwise disjoint subfamily of convex sets and a subfamily in which no two elements are comparable by any of the partial orders corresponds to a pairwise intersecting one.

Given any family $\mathcal{C}$ of convex sets in the plane, we superimpose these four partial orders on them. By Corollary 1.4, $\mathcal{C}$ has a subset of size at least $n^{1/5}$ which is either a chain in one of the partial orders, in which case this is a subset of pairwise disjoint sets and we are done. Or this set is an antichain for each of the partial orders, in which case it is a subset of pairwise intersecting sets. $\square$

We can show by an explicit construction given by Larman *et al.* [15] that $f(n) \leq n^{\frac{\log 2}{\log 5}}$. The construction relies on the following claim

Theorem 2.3. *Given points p and q in the plane, there exist five segments $p_1q_1, p_2q_2, \ldots p_5q_5$ such that the intersection graph of the segments is a 5-cycle. Also, each of the p_is is within an ϵ neighbourhood of p and each of the q_is is within an ϵ neighbourhood of q.*

The proof of the claim is simply by showing an explicit construction as seen in Figure 2.2. Note that the largest independent set as well as the largest clique in such a construction is of size two. It is easy to see that ϵ here can be made as small as required. Let $\mathcal{C}_1$ be a family of five such segments. Given $\mathcal{C}_i$, we define $\mathcal{C}_{i+1}$ as follows. Choose an ϵ_{i+1} small enough so that an ϵ_{i+1} neighbourhood of each of the end points of the segments in $\mathcal{C}_i$ intersects no segments other than the segment whose end point it is. To get $\mathcal{C}_{i+1}$ we replace each segment pq in $\mathcal{C}_i$ by the construction given in Claim 2.3. Clearly, $|\mathcal{C}_i| = 5^i$. Also, if two segments s_1 and s_2 are independent in $\mathcal{C}_i$ then all the segments in $\mathcal{C}_{i+1}$ which arose from s_1 will be independent to all those which arose from s_2. The same holds also for two segments which intersect in $\mathcal{C}_i$. This implies that the largest pairwise intersecting subset as well as pairwise independent subset in $\mathcal{C}_i$ is of size 2^i. So for n equal to 5^i, 2^i is equal to $n^{\frac{\log 2}{\log 5}}$. This proves that $f(n) \leq n^{\frac{\log 2}{\log 5}}$ as required. The best known upper bound is $f(n) \leq n^{0.405}$, due to Kynčl [14].

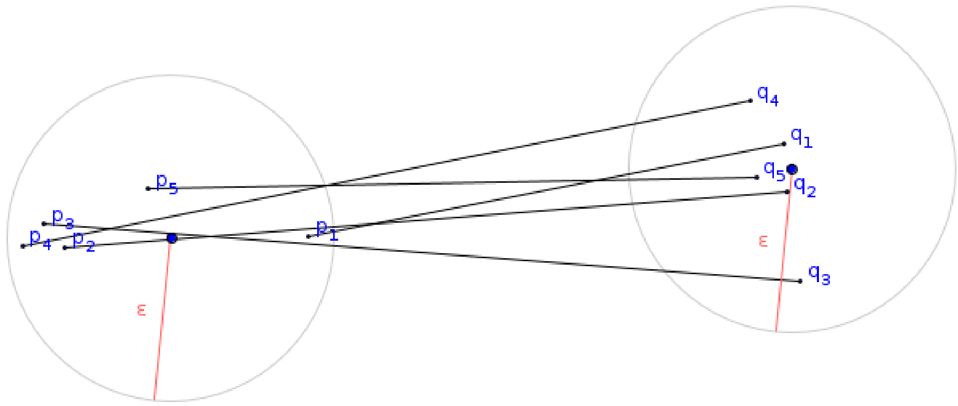

FIGURE 2.2: Each p_iq_i intersects with only $p_{i-1}q_{i-1}$ and $p_{i+1}q_{i+1}$ where the index is assumed cyclic over $1, 2, 3, 4, 5$

We remark that it was shown by Tietze [21] that every graph can be represented as the intersection pattern of a family of convex bodies in $\mathbb{R}^3$. So we cannot expect a superlogarithmic lower bound to hold for analogously defined problems for intersection graphs of convex bodies in higher dimensions.

2.1.2 Strong Erdős-Hajnal property

It turns out that the intersection graph of convex sets in the plane also has the strong Erdős-Hajnal property as shown by Fox *et al.* [10].

Theorem 2.4 (Fox *et al.* [10]). *If G is the intersection graph of n convex sets in the plane, then at least one of G or $\bar{G}$ has a bi-clique of size cn for a universal constant c.*

We first prove the preliminary lemmas and theorems, followed by a proof of the main result. We note that we replace one component of the original proof of Fox *et al.* [10] by a later result proved by Tomon [22] which is mentioned in Theorem 1.7, thereby shortening the proof a bit.

For two vertical parallel lines L_1 and L_2 in the plane, the region between the vertical lines defines a vertical strip. We say that a connected set α is a bridge between L_1 and L_2 if it intersects both L_1 and L_2. Let $y_1(\alpha)$ and $y_2(\alpha)$ denote the least y-coordinate at which α intersects L_1 and L_2 respectively. y_1 and y_2 are total orders on the set of bridges between L_1 and L_2, the intersection of which gives rise to a 2-dimensional partial order, say $\preceq_l$.

Lemma 2.5. *We are given a family A of bridges between two vertical lines L_1 and L_2 and a family B of at most ϵn connected sets lying in the strip between L_1 and L_2. Suppose the following two conditions hold*

1. *The intersection graph of $A \cup B$ has at least $25\epsilon^2 n^2$ edges.*
2. *The intersection graph of A has at most $4\epsilon^2 n^2$ edges.*

Then there exist $A'' \subseteq A$ and $B'' \subseteq B$ such that $|A''| = |B''| \geq \epsilon^2 n$ and every element of A'' intersects every element of B''.

Proof. Since B has at most ϵn elements, it can have at most $\epsilon^2 n^2$ intersecting pairs. We know that the number of intersecting pairs in $A \cup B$ is at least $25\epsilon^2 n^2$ and in B is at most $\epsilon^2 n^2$. Combined with the fact that the number of intersecting pairs in A is at most $4\epsilon^2 n^2$, this implies that there must be at least $20\epsilon^2 n^2$ intersecting pairs in $A \times B$.

Let $d = \epsilon n$ and A' be the set of bridges intersecting less than d other elements of A. So every element in $A \setminus A'$ intersects at least d other elements of A and therefore $|A \setminus A'| \leq (2.4\epsilon^2 n^2)/d \leq 8\epsilon n$. Therefore, the number of intersecting pairs in $A \setminus A' \times B$ is at most $|A \setminus A'||B| \leq 8\epsilon^2 n^2$. Since the number of intersecting pairs in $A \times B$ is at least $20\epsilon^2 n^2$, this implies that the number of intersecting pairs in $A' \times B$ is at least $12\epsilon^2 n^2$.

Let $B' \subseteq B$ so that each element in B' intersects at least d elements of A'. Then the number of intersecting pairs in $A' \times B'$ is at least $12\epsilon^2 n^2 - 5d|B| \geq 7\epsilon^2 n^2$.

We order the elements of A' by the total ordering y_1 and label them $1, 2, 3, \ldots |A'|$. We say that the distance between two elements of A' is the difference between their labels. We observe that if $\alpha_1, \alpha_2 \in A'$ intersect, and we have $\alpha \in A'$ such that $y_1(\alpha_1) \leq y_1(\alpha) \leq y_1(\alpha_2)$ then α must intersect one of α_1 or α_2. Since every element in A' intersects with at most $d-1$ other elements A, any two $\alpha_1, \alpha_2 \in A'$ at a distance of more than $2d-1$ cannot intersect, because if they did then there would be more than $d-1$ elements of A' between them that would intersect one of α_1 or α_2.

Now let $\beta \in B'$. Say β intersects $deg(\beta)$ elements of A'. Therefore there exist two elements α_1, α_2 of A' at a distance of at least $deg(\beta) - 1 \geq 5d - 1$ which intersect β. By our previous observation, these two elements of A' cannot intersect as they are too further apart. Also, there must be $deg(\beta) - 4d$ elements of A' between α_1 and α_2 which do not intersect α_1 or α_2 and therefore must intersect β (see Figure 2.3). We divide the ordered set $A' = \{1, 2, 3, \ldots |A'|\}$ into disjoint contiguous intervals

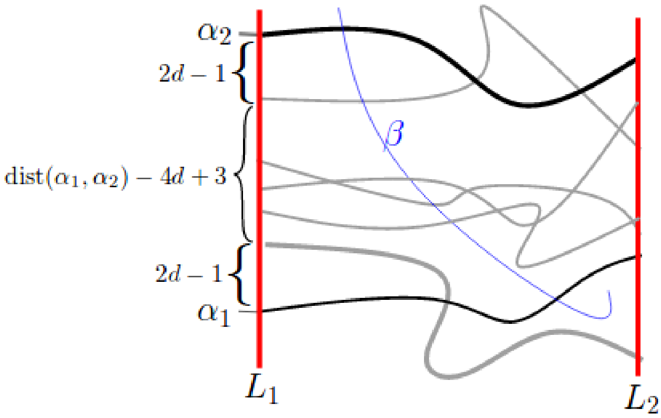

FIGURE 2.3: Figure from Fox *et al.* [10]. If β intersects α_1 and α_2 it must intersect every bridge which lies entirely between α_1 and α_2.

of size $d/2$. The number of intervals $s \leq \frac{|A'|}{d/2} \leq \frac{n}{\epsilon n/2} \leq \frac{2}{\epsilon}$. For each $\beta \in B'$ there are at least

$$\left\lfloor \frac{deg(\beta) - 4d}{d/2} \right\rfloor - 1 \geq \left\lfloor \frac{2}{d} deg(\beta) \right\rfloor - 9 > 0$$

intervals for which β intersects every bridge in each of those intervals. Summing up over all $\beta \in B'$ we get that there are a total of

$$\frac{2}{d} \sum_{\beta \in B'} deg(\beta) - 10|B'| \geq 14\epsilon n - 10\epsilon n = 4\epsilon n$$

intervals for which there is at least one $\beta \in B'$ which intersects every element of the interval. Therefore, by the pigeonhole principle there is an interval I and $B'' \subseteq B$ such that $|B''| \geq 4\epsilon n/s \geq$

$\epsilon^2 n$ such that every element of B'' intersects with every element of I. Let A'' be the elements of A' corresponding to the labels in I. $|A''| \geq d/2 = \epsilon n/2 \geq \epsilon^2 n$. □

We will also use the following result for convex bridges between lines. For a convex set α which is a bridge between L_1 and L_2, denote by $s(\alpha)$ the line segment joining the points with the lowest y-coordinates at which α intersects L_1 and L_2. Let the *lower curve* $l(\alpha)$ of α be the part of α below $s(\alpha)$ and the *upper curve* $u(\alpha)$ be the part of α above $s(\alpha)$ which is between L_1 and L_2.

Lemma 2.6. *Let A be a family of n convex bridges between L_1 and L_2. If there exists $\epsilon > 0$ such that the intersection graph G_c of the convex bridges has at least $\epsilon^2 n$ edges, then G_c contains a bi-clique of size at least $\epsilon n/6$.*

Proof. We divide the set of intersection pairs of A into 5 colour classes. Suppose α_1, α_2 is an intersecting pair. We colour the pair according to the number of the first condition from the following, which applies to it

1. α_1 or α_2 intersect along L_1 or L_2.
2. $s(\alpha_1)$ and $s(\alpha_2)$ intersect
3. $l(\alpha_1)$ and $l(\alpha_2)$ intersect
4. $u(\alpha_1)$ and $u(\alpha_2)$ intersect
5. $l(\alpha_1)$ and $u(\alpha_2)$ intersect or $u(\alpha_1)$ and $l(\alpha_2)$ intersect

Figure 2.4 shows examples of intersections pairs of each colour.

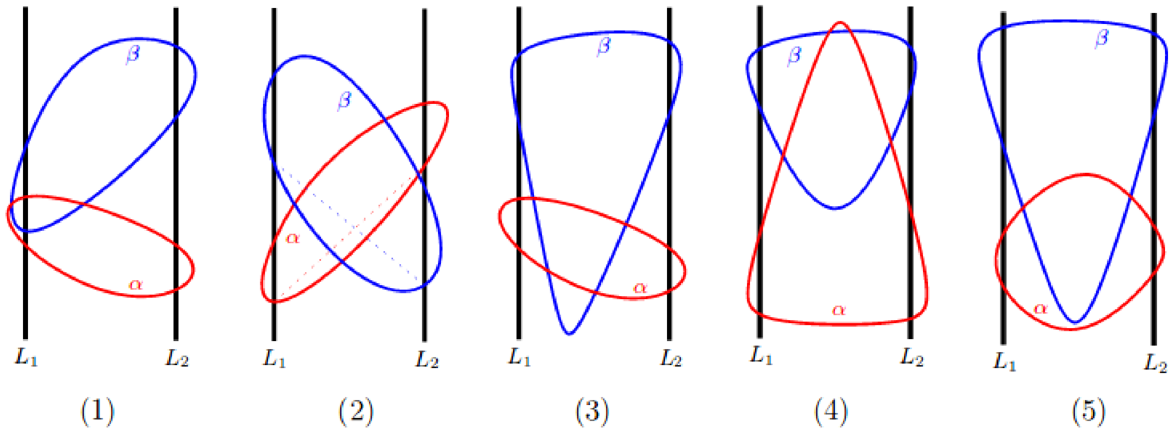

FIGURE 2.4: Figure from Fox *et al.* [10]. Examples of constructions with colouring 1, 2, 3, 4 and 5.

We divide the problem into five cases, each corresponding to having a lower bound on the number of intersection pairs of each of the colour classes.
Case 1: There are at least $\epsilon^2 n/3$ pairs with colour 1.
At least $\epsilon^2 n/6$ of these pairs intersect on one of the vertical lines, say L_1. So we have a family of n intervals on a line with $\epsilon^2 n/6$ intersecting pairs of intervals. Therefore the intersection graph of the intervals has to have a clique of size $\epsilon n/6$, because if not then it can be shown to be $(\epsilon n - 2)$-degenerate, which contradicts the fact that there are at least $\epsilon^2 n/6$ intersecting pairs of intervals.
Case 2: There are at least $\epsilon^2 n/6$ pairs with colour 2.
By the definition of the 2-dimensional partial order $\preceq_l$, every intersecting pair with colour 2 is incomparable under $\preceq_l$. Theorem 1.6 implies that there is a bi-clique of size $\epsilon n/6$ with each intersecting pair having colour 2.
Case 3: There are at least $\epsilon^2 n/6$ pairs with colour 3.

For any $\alpha \in A$, we denote by $C_3(\alpha)$ the set of convex bridges $\beta \in A$ such that $\alpha \preceq_l \beta$ and the pair (α, β) has colour 3. Any $\beta_1, \beta_2 \in C_3(\alpha)$ intersect. If at least $\epsilon n^2/6$ pairs have colour 3 then there exists $\alpha_0 \in A$ for which $|C_3(\alpha_0)| \geq \epsilon n/6$ and therefore we have a clique of size $\epsilon n/6$ in G.
For *Case 4* and *Case 5* we define analogous sets C_4 and C_5 respectively and argue as in *Case 3*. □

We prove the following theorem, which is essential to the proof of the main theorem.

Theorem 2.7. *Let S be the intersection graph of n convex sets in the plane. If $|E(S)| \geq \delta n^2$ for some $\delta > 0$, then there exists a constant c depending only on δ such that S contains a bi-clique of size $\geq c(\delta)n$.*

Proof. We assume without loss of generality that S is a family of convex polygons, because we can always find a family of convex polygons which has the same intersection graph as a family of convex sets in the plane. We also assume without loss of generality that the set of $2n$ points forming the left and right end-points of the convex polygons have distinct x-coordinates. If not we can perturb the points just a little bit without changing the intersection graph. We now divide the plane into strips and use the lemmas for bridges and connected sets between strips that we have. Towards this purpose, we select a point inside the intersection of each intersecting pair of polygons that we have in the family. We call such points the special points.

We divide the plane into $\Delta = 10/\delta + 1$ strips so that each strip contains at most $2n/\delta \leq \delta n/4$ extreme points. Note that the rightmost and the leftmost strips are just half-planes. Note that the half-planes on both the extremes can have at most $\binom{\delta n/4}{2} \leq \delta^2 n^2/32$ special points, because they cannot intersect with polygons for which none of the extreme points lie in that half-plane. By the pigeonhole principle we have that at least one of the strips must have at least $\delta n^2/\Delta \geq \delta^2 n^2/11$ special points. This cannot be a half-plane because it has at most $\delta^2 n^2/32$. Therefore this $\delta^2 n^2/11$ special points must lie between two vertical lines L_1 and L_2, which for the vertical strip R. We will apply lemma 2.5 and lemma 2.6 to this strip to prove the result. Towards this purpose we clip the convex polygons in R. Then let A be the set of polygons forming a bridge between L_1 and L_2 and B be the remaining set of polygons. B can have at most as many polygons as the number of extreme points in R, which is at most $\delta n/4$ by construction.

In case we have more than $\delta^2 n^2/100$ intersecting pairs in A, then we can use Lemma 2.6 to get that the intersection graph of A and therefore the intersection graph of S has a bi-clique of size at least $\delta^2 n^2/600$. Note that for the proof of the theorem we only looked at intersections occurring inside R. We can still safely use this theorem because we still have the same bi-clique in the intersection graph of A if sets in A do not intersect within R but intersect outside it.

In the other case if we have less than $\delta^2 n^2/100$ intersecting pairs in A. We know that we have at least $\delta^2 n^2/11$ intersecting pairs in $A \cup B$. So we have all the conditions required to apply Lemma 2.5, which gives us a bi-clique of size $c(\delta)n$ in $A \cup B$. □

We can finally give a proof of the main theorem.
Proof of Theorem 2.4: Let G be an intersection graph of a set S of n convex sets in the plane. We assume again that it is the intersection graph of n convex polygons and that the $2n$ extreme points of the polygons have distinct x-coordinates.

First, suppose that no vertical line L intersects more than $n/3$ sets in S. Then we choose a line L which does not contain any of the $2n$ extreme points and exactly $n/3$ polygons lie entirely in the open half-plane to the left of L. Since L intersects at most $n/3$ sets in S, we also have $n/3$ sets entirely in the open half-plane to the right of L. Since each set entirely on the right of L is disjoint to each set entirely on the left of L, we have a bi-clique of size $2n/3$ in $\bar{G}$.

In the other case, there is a line L which intersects at least $n/3$ sets in S. We consider $S' \subseteq S$ of the the sets intersecting L. Let G_L be the intersection graph of S'. Suppose we have more than δn^2 edges in G_L for some $\delta > 0$, then by Theorem 2.7 we have a bi-clique of constant times n in G_L for a constant depending only on δ. In the other case we have at least $1 - \delta n^2$ edges in the disjointness graph of S'. Note that the disjointness graph of S' is exactly the union of the comparability graph of the four partial orders defined on convex sets in Definition 2.2. Therefore we can use Theorem 1.7 with $r = 4$ and appropriate value of ϵ to get that the disjointness graph of S' has a bi-clique of size at least constant times n with the constant depending only on δ. Once we have fixed an appropriate delta, it implies the theorem. $\square$

2.2 Axis-parallel rectangles in the plane

We can get a better bound than the one in Theorem 2.1 if we restrict ourselves to an even simpler family of graphs, namely the intersection graphs of axis-parallel rectangles in the plane.

Theorem 2.8 (Larman *et al.* [15]). *Let $r(n)$ be the largest number with the property that the intersection graph of any n axis-parallel rectangles in the plane contains either an independent set or clique of size $r(n)$. Then $r(n) \geq \sqrt{\frac{n}{2\log n}}$.*

Proof. Denote by $\phi(m)$ the smallest number n for which the intersection graph of any family of n axis-parallel rectangles in the plane has $\alpha(G)\omega(G) \geq m$. Proving an upper bound on $\phi(m)$ proves a lower bound on $\alpha(G)\omega(G)$ for a fixed n and therefore proves a lower bound on $max\{\omega(G), \alpha(G)\}$ as required. We will show by induction on k that $\phi(2^k) \leq k2^k$. Substituting $k = \lfloor \log n - \log\log n \rfloor$ we get that $\phi(2^k) \leq n$ and $2^k \geq \frac{n}{2\log n}$. This implies that for n rectangles $\alpha(G)\omega(G) \geq \frac{n}{2\log n}$ and therefore $max\{\omega(G), \alpha(G)\} \geq \sqrt{\frac{n}{2\log n}}$ as required. So it only remains to prove that $\phi(2^k) \leq k2^k$.

For $k = 1$, it is easy to see that $\phi(2) = 2$. Now suppose it holds for $k-1$, that is $\phi(2^{k-1}) \leq (k-1)2^{k-1}$. Then we want to show that it holds for a family $\mathcal{R}$ of $k2^k$ rectangles. We assume without loss of generality that the right vertical side of each of the rectangles in $\mathcal{R}$ is distinct. Let l_0 be the $(k-1)2^{k-1}$th such right vertical line from the left. Denote by $\mathcal{R}_0$ the family of rectangles in $\mathcal{R}$ that intersect l_0. First, suppose $|\mathcal{R}_0|$ is greater than or equal to 2^k. Note that two rectangles in $\mathcal{R}_0$ intersect if and only if the intervals of intersection of the corresponding rectangles with l_0 are overlapping. By the result on intervals on a line mentioned previously, this implies that $\alpha(\mathcal{R}_0)\omega(\mathcal{R}_0) \geq 2^k$. We are done since $\alpha(\mathcal{R})\omega(\mathcal{R}) \geq \alpha(\mathcal{R}_0)\omega(\mathcal{R}_0)$.

So we assume that $|\mathcal{R}_0| < 2^k$. Denote by $\mathcal{R}_l$ the family of rectangles in $\mathcal{R}$ strictly to the left of l_0. $|\mathcal{R}_l| = (k-1)2^{k-1}$ by definition of l_0. Denote by $\mathcal{R}_r$ the family of rectangles in $\mathcal{R}$ strictly to the right of l_0. Then $|\mathcal{R}_r| = |\mathcal{R}| - |\mathcal{R})_0| - |\mathcal{R}_l| \geq k2^k - 2^k - (k-1)2^{k-1} \geq (k-1)2^{k-1}$. Now the size of the largest independent set in $\mathcal{R}$ is at least the sum of size of largest independent sets in $\mathcal{R}_r$ and $\mathcal{R}_l$. So we have

$$\alpha(\mathcal{R})\omega(\mathcal{R}) \geq (\alpha(\mathcal{R}_l) + \alpha(\mathcal{R}_r))max\{\omega(\mathcal{R}_l), \omega(\mathcal{R}_r)\} \geq \alpha(\mathcal{R}_l)\omega(\mathcal{R}_l) + \omega(\mathcal{R}_r)\alpha(\mathcal{R}_r) \geq 2^k$$

This completes the proof. $\square$

3 Geometric graphs

A geometric graph $G = (V, E)$ is a graph defined in the plane so that V corresponds to a set of points on the plane in general position and each edge is a closed line segment connecting the two points on the plane corresponding to the vertices of that edge.

For a geometric graph G on n vertices, we define $e_k(n)$ to be the smallest number such that if G has more than $e_k(n)$ edges, then it must have $k+1$ pairwise disjoint edges. The question of determining $e_k(n)$ was first raised by Avital and Hanani [3], and Kupitz and Perles [13].

3.1 A simple bound on $e_k(n)$

Using the same partial orders as defined for convex sets in Section 2.1.1, where we think of each closed line segment as a convex compact set, Pach and Tőröcsik [19] proved the following theorem.

Theorem 3.1. *Any geometric graph G with n vertices and number of edges $m > k^4 n$ has $k+1$ pairwise disjoint edges. That is, $e_k(n) \leq k^4 n$.*

Proof. We can consider each edge, which is a line segment, as a convex set in the plane. Let $\preceq_1$, $\preceq_2$, $\preceq_3$ and $\preceq_4$ be the same partial orders as defined in Definition 2.2; on the set of edges of the graph. Let G be a geometric graph with n vertices and m edges so that it has no $k+1$ pairwise disjoint edges. This implies that it does not have a chain of length $k+1$ in any of the partial orders. Therefore by Dilworth's Theorem, for each of the partial orders $\preceq_i$, we can decompose the set of edges into at most k classes so that no two edges in the same class are comparable by $\preceq_i$. Superimposing these classes for all the four partial orders, we get a decomposition of the set of edges into at most k^4 classes so that no two edges in the same class are comparable by any of the four partial orders. By the property of the partial orders, each edge is any of the k^4 classes must intersect with every other edge in that class. So the number of edges in each class is at most $e_1(n)$, which is known to be equal to n by a result of Erdős [7]. Therefore, the total number of edges in G is at most $k^4 n$. $\square$

3.2 An improved bound on $e_k(n)$

Tóth [23] used the same partial orders in a slightly more involved proof to show the following

Theorem 3.2. *Any geometric graph G with n vertices and number of edges $|E(G)| = m > 2^9 k^2 n$, has $k+1$ pairwise disjoint edges. That is, $e_k(n) \leq 2^9 k^2 n$.*

Note that we again have the same four partial orders defined on the segments representing the edges of G in the plane. We show that if there does not exist a chain of length $k+1$ in any of the four partial orders then $m \leq 2^9 k^2 n$. This implies that if $m > 2^9 k^2 n$ then there is a chain of length $k+1$ in one of the partial orders and therefore a set of $k+1$ pairwise disjoint edges which is what we need in Theorem 3.2. We will use the following elementary theorem in the proof.

Lemma 3.3. *Given a sequence of real numbers $A = \{a_1, a_2, \dots a_m\}$, there exist disjoint monotone subsequences $A_1, A_2, \dots A_r$ such that $|A_i| = \lceil\sqrt{\frac{m}{2}}\rceil$ for each i and $|A_1| + |A_2| + \dots + |A_r| \geq \frac{m}{2}$*

Proof. This can be proved by defining a partial order on A as follows. We say that $a_i \preceq a_j$ if $i \leq j$ and $a_i \leq a_j$. Then by Theorem 1.3 every set $S \subseteq A$ of size at least $m/2$ has a chain or an antichain of length $\lceil\sqrt{\frac{m}{2}}\rceil$. Note that a chain corresponds to an increasing subsequence and an antichain corresponds to a decreasing subsequence. Let this monotone subsequence of length $\lceil\sqrt{\frac{m}{2}}\rceil$ be A_1. We can then get a monotone subsequence A_2 of length $\lceil\sqrt{\frac{m}{2}}\rceil$ from $A \setminus A_1$, A_3 from $A \setminus \{A_1 \cup A_2\}$ and so on, as long as the number of remaining elements is at least $\frac{m}{2}$. We stop when the number of remaining elements in A goes below $\frac{m}{2}$ and therefore the sum of number of elements in the subsequences is at least $\frac{m}{2}$ as required. □

We assume without loss of generality that the x-coordinate of no two vertices of G coincide. We label the vertices in the increasing order of their x-coordinates as $v_1, v_2, \dots v_n$. We say that an edge v_iv_j is a right (left) edge of v_i if $j > i$ ($j < i$). We call the number of right (left) edges of v_i as the right (left) degree of v_i and denote it by r_i (l_i).

Left and right decomposition of G: For a vertex $v = v_i$ with right degree $r = r_i$, let $e_1e_2, \dots e_r$ be the right edges of v ordered clockwise. We decompose the sequence $x(e_1), x(e_2), \dots x(e_r)$ as in Lemma 3.3 to get several monotone subsequences. We remove the edges which are not part of any of the monotone subsequences. We call each such subsequence a *right-block* of edges. We say that an edge e which remains is *right-increasing* if the right block it is part of is an increasing subsequence and *right-decreasing* otherwise. We do such a decomposition for every vertex v. We assume without loss of generality that at least half of the remaining edges are right-increasing, and delete all the right-decreasing edges to obtain the graph G'. Note that we delete at most half of the edges of G when we remove edges which are not part of any of the monotone subsequences and at most half of those remaining when we remove right-decreasing edges. So we have $|E(G')| \geq \frac{1}{4}|E(G)|$. We then analogously do a left decomposition of every vertex in G by looking at all the left edges of a vertex in anti-clockwise order. We again assume without loss of generality that we have more left-increasing edges than left-decreasing edges and so we delete the left-decreasing edges. We will justify that this is without loss of generality in Lemma 3.4. This gives us a graph G'' such that

$$|E(G'')| \geq \frac{1}{4}|E(G')| \geq \frac{1}{16}|E(G)| \tag{3.1}$$

We say that an edge $e_2 = v_iv_{j_2}$ in G'' is a *right-zag* of $e_1 = v_iv_{j_1}$ if both e_1 and e_2 are right edges of v_i and e_2 immediately follows e_1 in the same right block of v_i. Analogously, we say that an edge $e_2 = v_iv_{j_2}$ is a *left-zag* of $e_1 = v_iv_{j_1}$ if both e_1 and e_2 are left edges of v_i and e_2 immediately follows e_1 in the same left block of v_i. We call a path $e_1e_2 \dots e_l$ a *zig-zag path* if the edges are alternatively right-zags and left-zags of the previous edges. More precisely, for any i if e_i is a right zag of e_{i-1} then e_{i+1} is a left zag of e_i and if e_i is a left zag of e_{i-1} then e_{i+1} is a right zag of e_i. Every path of length one is trivially a zig-zag path.

We bound the number of edges in G'' in terms of k by bounding the length of maximal zig-zag paths in G'' and then bounding the number of maximal zig-zag paths.

Lemma 3.4. *The length of any maximal zig-zag path in G'' can be at most $2k$.*

Proof. It is sufficient to show that if we have a zig-zag path $e_1e_2\dots e_l$ in G'', then $e_{i+2} \preceq_3 e_i$ for every i upto $l-2$. Let $e_i = v_1v_2$, $e_{i+1} = v_2v_3$ and $e_{i+2} = v_3v_4$.

The **first case** is when e_{i+1} is a right zag of e_i and e_{i+2} is a left zag of e_{i+1}. Since e_{i+1} is a right zag of e_i we have $x(v_3) \geq x(v_1)$ and since e_{i+2} is a left zag of e_{i+1} we have $x(v_4) \geq x(v_2)$. Since v_1 and v_2 are the left and right endpoints of e_i respectively and v_3 and v_4 are the left and right endpoints of e_{i+2} respectively, this implies that the right endpoint of e_{i+2} is to the right of the right endpoint of e_i and the left endpoint of e_{i+2} is to the right of the left endpoint of e_i. It is clear that e_{i+2} is below e_i. Therefore by our definition of partial orders we have $e_{i+2} \preceq_3 e_i$. So if $l \geq 2k+1$ then we have a chain of length $k+1$ in $\preceq_3$, which contradicts our assumption that there is no chain of length $k+1$ in any of the partial orders. Therefore $l \leq k$.

The **second case** when e_{i+1} is a left zag of e_i and e_{i+2} is a right zag of e_{i+1} has an analogous proof as to the first case.

If we had some combination of edges other than 'right-increasing' and 'left-decreasing', we would just have used a different partial order to prove this theorem. Everything else remains the same. Therefore the result is without loss of generality as claimed earlier. □

It only remains to bound the number of maximal zig-zag paths in G''.

Lemma 3.5. *There are at most $\sqrt{\frac{2E(G)}{n}}$ maximal zig-zag paths in G''.*

Proof. For any vertex v the number of maximal zig-zag paths which can start at that vertex is at most the number of blocks at v. The number of right blocks at v is at most $\sqrt{\frac{r}{2}}$ and the number of left blocks at v is at most $\sqrt{\frac{l}{2}}$. So the total number of maximal paths is bounded by

$$\sum_{i=1}^{n}\left(\sqrt{\frac{r_i}{2}}+\sqrt{\frac{l_i}{2}}\right) \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}\frac{r_i+l_i}{2}} = \sqrt{2|E(G)|n}$$

□

Proof of Theorem 3.2: To bound the number of edges in G'' we note that each edge in G'' can be part of at most one left block and at most one right block, therefore each edge has at most one right-zag and one left-zag and also that each edge is a right-zag and a left-zag of at most one edge. So each edge in G'' can be a part of at most two maximal zig zag paths in G''. We now proceed to bound the number of edges in each maximal zig-zag path in G''. Combining Lemma 3.4 and Lemma 3.5 with Equation 3.1 we get that

$$\frac{1}{16}|E(G)| \leq |E(G'')| \leq \frac{2k\sqrt{2|E(G)|n}}{2}$$

This implies $|E(G)| \leq 2^9k^2n$. □
It remains open whether we can improve this bound to $\mathcal{O}(kn)$.

We now see some results about the χ-boundedness of disjointness graphs of convex sets.

4 Colouring properties of disjointness graphs

We say that a family of graphs $\mathcal{F}$ is χ-bounded by a function f if for every G in $\mathcal{F}$ we have $\chi(G) \leq f(\omega(G))$. χ-boundedness can be seen as a kind of generalization of perfectness. It was shown by Asplund *et al.* [2] that the family of intersection graphs of axis-parallel rectangles in the plane is χ-bounded by $\mathcal{O}(x^2)$. On the other hand, it was shown by Burling [5] that there is a family of triangle-free intersection graphs of axis parallel boxes in $\mathbb{R}^3$ with unbounded chromatic number, hence showing that there is no functional relationship between f and ω for the family of intersection graphs of axis-parallel boxes in $\mathbb{R}^3$.

We know that the complement of every perfect graph is perfect, as proved by Lovász [16]. We might be tempted to believe that such a result might also hold for χ-boundedness, in the sense that the family of complements of graphs from a χ-bounded family is also χ-bounded. It turns out that this is not so. It was shown by Károlyi [12] that the family of disjointness graphs of axis parallel boxes in $\mathbb{R}^d$ is χ-bounded, in contrast to the result by Asplund *et al.* [2] mentioned above. Also, it was shown by Pawlik *et al.* [20] that the family of intersection graphs of segments in the plane is not χ-bounded, while on the other hand the family of disjointness graphs of segments in the plane is, as we will see next.

It follows from the partial orders defined on the convex sets in the plane in Definition 2.2 that the disjointness graph of convex sets in the plane is χ-bounded.

Theorem 4.1. *The disjointness graph G of a family of convex sets in the plane is χ-bounded by $f(x) = x^4$.*

Proof. We know by Dilworth's theorem that the comparability graph of a partial order is perfect. The disjointness graph of G is nothing but the union of the comparability graphs of the four partial orders we defined in Definition 2.2. Clearly the comparability graph of none of the four partial orders has a clique larger than $\omega(G)$, therefore each of them can be coloured by at most $\omega(G)$ colours. Therefore the union of the four graphs can be coloured by at most $(\omega(G))^4$ colours. $\square$

4.1 Segments in space

Having seen that the family of disjointness graphs of segments in the plane is χ-bounded, the natural question is whether the family of disjointness graphs of segments in higher dimensions is χ-bounded. The question is answered in the affirmative by Pach *et al.* [18].

Theorem 4.2. *The disjointness graph of segments in $\mathbb{R}^d$ is χ-bounded by $f(x) = x^3 + x^4$*

Before proving the theorem we will show that if the segments are restricted to k planes in $\mathbb{R}^d$ then their disjointness graph is χ-bounded.

Lemma 4.3. *Let G be the disjointness graph of a family of segments in $\mathbb{R}^d$ such that each of the segment lies on one of k two dimensional planes. Then G is χ-bounded by $f(x) = (k-1)x + x^4$.*

Proof. Denote by $\pi_1, \pi_2, \ldots \pi_k$ the planes on which the segments lie. We use the same notation to denote the segment as well as the vertex it represents. We partition the segments into $V_1, V_2, \ldots V_k$ so that a segment l is in V_i if i is the highest index for which $l \in \pi_i$.

We recursively define sets Z_i, W_i for $i \in [k]$. Let $W_1 = V_1$ and Z_1 be the largest clique in $G[W_1]$. Suppose now that we have already defined $W_1, W_2, \ldots W_{i-1}$ and $Z_1, Z_2, \ldots Z_{i-1}$. Then we define W_i to be composed of every element v of V_i such that v is adjacent to every vertex in $\cup_{j=1}^{i-1} Z_j$ and let Z_i be the largest clique in $G[W_i]$. Note that with this definition $Z = \cup_{j=1}^{k} Z_j$ is a clique in G. Therefore we have $|Z| \le \omega(G)$.

We say that a point p on a segment $s \in V_i$ is a piercing point if s intersects π_j at p for some $j > i$. Clearly, s can pierce π_j only at one point. We define $V_0 = G \setminus \{W_1 \cup W_2 \ldots W_k\}$. Suppose we have a segment $s \in V_i \setminus W_i$, then s must contain a piercing point. This is because if $s \notin W_i$ then it implies that there is at least one segment t in $\cup_{j=1}^{i-1} Z_j$ that s intersects. Then the point at which t intersects s is a piercing point for t.

We associate a colour with each piercing point and colour the segments in V_0 by giving them the colour of one of the piercing points which lie on it. This is clearly a valid colouring of V_0 because no two disjoint segments in V_0 can possibly share a piercing point and therefore the same colour. It just remains to bound the number of piercing points.

We have seen that each segment in V_0 has a piercing point that lies on a segment in $|Z|$. And each segment in Z_i can have at most $k - i$ piercing points. If we denote by P the set of piercing points that lie on segments in Z, then we have

$$|P| \le \sum_{i=1}^{k} (k-i)|Z_i| \le (k-1)|Z| \le (k-1)\omega(G).$$

Since we already saw that it is sufficient to look at piercing points which lie on one of the segments in Z to obtain a proper colouring of V_0, this implies that

$$\chi(G[V_0]) \le (k-1)\omega(G) \tag{4.1}$$

Since all the segments in a particular W_i lie in a single plane, we can colour them by Theorem 4.1, which implies that for each $i \in [k]$ we have

$$\chi(G[W_i]) \le (\omega(G[W_i]))^4 \tag{4.2}$$

By definition V_0 and W_i are disjoint for each $i \in [k]$. Therefore we have

$$\chi(G) \le \chi(G[V_0]) + \sum_{i=1}^{k} \chi(G[W_i]).$$

Plugging in Equation 4.1 and Equation 4.2 in the above equation, we get

$$\chi(G) \le (k-1)\omega(G) + \sum_{i=1}^{k} (\omega(G[W_i]))^4 \le (k-1)\omega(G) + (\omega(G))^4.$$

□

We can now prove the main theorem.

Proof of Theorem 4.2: We are given a family of segments in $\mathbb{R}^d$ with disjointness graph G. We extend each segment in the family to a line in $\mathbb{P}^d$ and look at this family of lines. Let G' be the disjointness graph of this family of lines in $\mathbb{P}^d$. Note that by Theorem 4.5 we have

$$\chi(G) \leq (\omega(G'))^2 \leq (\omega(G))^2.$$

Suppose we have an optimal colouring of G' using $\chi(G')$ colours. Then all the lines which have the same colour are mutually intersecting. Therefore, they either all lie on the same plane or pass through the same point. If all the lines corresponding to a colour i lie on the same plane then we call such a colour a 'planar' colour. Otherwise, we call it a 'pointed' colour. Suppose we have k planar colours and $\chi(G') - k$ pointed colours. Now if we consider the graph $G_0 \subseteq G$ of the segments corresponding to the lines with planar colours, then the segments in G_0 clearly lie in a union of k planes. This implies by Lemma 4.3 that

$$\chi(G_0) \leq (k-1)\omega(G_0) + (\omega(G_0))^4 \leq (k-1)\omega(G) + (\omega(G))^4 \tag{4.3}$$

Suppose G_i is the disjointness graph of the segments corresponding to the lines which were given the i^{th} pointed colour. Then clearly G_i cannot have an induced cycle of length greater than three because all the segments in G_i which intersect must pass through the same point. Therefore G_i is the complement of a chordal graph, which by a result of Hajnal *et al.* [11] is known to be perfect. Therefore we have

$$\chi(G_i) = \omega(G_i) \leq \omega(G) \tag{4.4}$$

Note that the G_is and G_0 are all mutually disjoint. Therefore we have

$$\chi(G) \leq \chi(G_0) + \sum_{i=1}^{\chi(G')-k} \chi(G_i).$$

Substituting Equation 4.3 and Equation 4.4 into the above equation, we get

$$\begin{aligned}
\chi(G) &\leq \chi(G_0) + \sum_{i=1}^{\chi(G')-k} \chi(G_i) \\
&\leq (k-1)\omega(G) + (\omega(G))^4 + \sum_{i=1}^{\chi(G')-k} \omega(G) \\
&\leq (k-1)\omega(G) + (\omega(G))^4 + (\chi(G') - k)\omega(G) \\
&\leq \chi(G')\omega(G) + (\omega(G))^4 \\
&\leq (\omega(G))^3 + (\omega(G))^4.
\end{aligned}$$

This completes the proof. □

4.2 Lines in space

After segments, we now move on to look at lines in higher dimensions. We denote the d-dimensional projective space by $\mathbb{P}^d$. We show that the disjointness graph of lines in $\mathbb{P}^d$ is χ-bounded, as is the disjointness graph of lines in $\mathbb{R}^d$. We first prove the result for lines in the projective space. We first prove a lemma that will be useful in proving the theorem.

Lemma 4.4. *Let G be the disjointness graph of lines in $\mathbb{P}^d$. If G has an isolated vertex v_0 then G is perfect.*

Proof. We construct a bipartite graph B from the given lines. We will show that the graph G is the line graph of this bipartite graph B and hence perfect. Let l_0 be the line which corresponds to the isolated vertex in G. Note that l_0 must intersect every other line $l \in V(G)$. Let the vertex set of the graph B be $I \cup P$, where I is the set of points on l_0 at which any other line intersects it and P is the set of planes in $\mathbb{P}^d$ which contain l_0 as well as any other line l from the family. For every line $l \in V(G) \setminus \{l_0\}$ we have an edge e_l in B, connecting the point $p \in I$ at which l_0 intersects l and the plane $P_l \in P$ which is spanned by l and l_0. Note that this might be a multigraph as we might have two lines l, l' such that l, l' and l_0 are on the same plane and l and l' intersect l_0 at the same point.

The crucial observation is that two lines $l, l' \in V(G) \setminus \{l_0\}$ intersect if and only if they intersect at some point on l_0 or they lie on the same plane. This implies that $l, l' \in V(G) \setminus \{l_0\}$ intersect if and only if e_l and $e_{l'}$ share a vertex. Therefore G is the line graph of a bipartite multigraph and is therefore perfect and it is know that line graphs of bipartite multigraphs are perfect. So G is obtained by adding the isolated vertex v_0 to a perfect graph and is therefore also perfect. □

We finally look at the family of disjointness graphs of lines in projective space.

Theorem 4.5. *The family of disjointness graphs of lines in $\mathbb{P}^d$ is χ-bounded by $f(x) = x^2$*

Proof. Let G be a disjointness graph of lines in $\mathbb{P}^d$. Let $C \subseteq V(G)$ be a maximal clique in G. Since the clique is maximal, for every $l \in V(G) \setminus C$ there exists a $c \in C$ which it is not adjacent to. This means that there is a partition of $V(G)$ into disjoint sets V_c for $c \in C$ such that $c \in V_c$ and c is isolated in $G[V_c]$. Therefore

$$\chi(G) \leq \sum_{i=1}^{|C|} \chi(G[V_c]) = \sum_{i=1}^{|C|} \omega(V_c) \leq |C|\omega(G) \leq (\omega(G))^2$$

where the second equality follows from Lemma 4.4. □

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